A PLAUSIBLE APPROXIMATION METHOD FOR LINEAR SYSTEMS AND ITS APPLICATION FOR THE DESIGN OF LOW-ORDER CONTROLLERS

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Abstract

The contribution of this paper is twofold. First, a plausible method for the approximation of a system's transfer function is introduced. Second, a procedure for designing a low-order controller based on this approximation method is presented.

The method proposed in this paper performs an order-reduction while considering the properties of the closed loop system. This ensures that the reduced-order controller achieves a performance close to the one of a full-order controller. Roughly speaking, the proposed procedure computes the low-order controller by approximating an equivalent closed-loop state feedback system and then performing an inverse operation.

1. Introduction

The problem of the design of low-order controllers has been addressed from many different angles. Today, this problem is usually divided in two subproblems. A common method consists in reducing the process model, and in a second step, in applying the analytical design method to the reduced model. Another method proposes to reduce a posteriori the order of the obtained controller. However, both ways are not optimal because the design of a controller and its reduction are coupled and should be solved simultaneously. Trying a direct approach, Hyland and Bernstein [1] derived the necessary conditions for an optimal fixed-order controller. Unfortunately, their approach involves the solution of four (two nonlinear) coupled matrix equations, and results, in general, in a hard computational problem.

This paper offers an alternative approach for the design of low-order controllers for SISO systems. (The MIMO case is currently under development.) The goal of order-reduction is to obtain a simplified control system, consisting of the reduced-order controller and of the full-order process, that meets the closed-loop specification. We suppose here that the specification is given as a required phase margin for stability. To achieve this, the proposed design method approximates an LQ-optimal loop-gain. An LQ-optimal loop-gain exhibits a phase margin $\varphi \geq 60$ degrees, and is minimum-phase. For a good approximation of an LQ-optimal loop-gain (minimum-phase property) both the poles and the zeros have to be accurately approximated. The well-known reduction methods however, e.g. the modal methods (Davison, [2]) and the famous method via balanced realization of Moore [3] where the Grammians are integrated over an infinite time horizon, suffer from the fact that they tend to approximate the poles better than the zeros; a stable system remains stable whereas a minimum-phase system may become a nonminimum-phase system. Therefore, we use the approximation method proposed by Brunner [4] which treats poles and zeros in a more balanced way. Based on this approximation method together with LQ-theory, the actual controller design procedure is presented in section 3..

2. The Approximation Method

In this section the so-called "closed-loop reduction" method, is presented. The following idea is suggested for obtaining a "balanced" approximation of the poles and the zeros, or of the denominator and the numerator respectively. Let $G(s) = z(s)/n(s)$ be the SISO-system to be reduced, n its order, and k the approximation-parameter. Instead of $G(s)$ we consider the feedback system with transfer function

$$H(s) = \frac{kG(s)}{1 + kG(s)} = \frac{kz(s)}{n(s) + kz(s)} \tag{1}$$

and reduce it by applying a modal or Moore's reduction method (see figure 1). The result is a reduced transfer function $H_{\mathrm{rd}}(s)$ of order $n_{\mathrm{rd}} < n$:

$$H_{\mathrm{rd}}(s) = \frac{z_{\mathrm{rd}}(s)}{n_{\mathrm{rd}}(s)} \tag{2}$$

On the other hand, $H_{\mathrm{rd}}(s)$ can be thought of as the transfer function of an analogous closed-loop system consisting of a system with the transfer function $G_{\mathrm{ap}}(s) = z_{\mathrm{ap}}(s)/n_{\mathrm{ap}}(s)$ and a constant controller k (see figure 1). Thus, the following condition holds:

$$H_{\mathrm{rd}}(s) = \frac{kG_{\mathrm{ap}}(s)}{1 + kG_{\mathrm{ap}}(s)} = \frac{kz_{\mathrm{ap}}(s)}{n_{\mathrm{ap}}(s) + kz_{\mathrm{ap}}(s)} \tag{3}$$

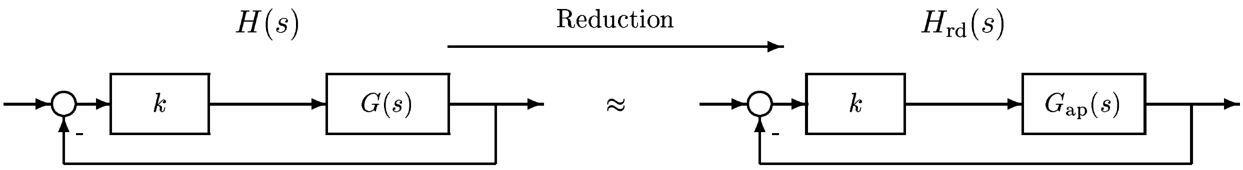

Figure 1: closed-loop reduction (k : approximation-parameter)

Equating equations (2) and (3) yields

$$\frac{z_{\rm rd}(s)}{n_{\rm rd}(s)} = \frac{k z_{\rm ap}(s)}{n_{\rm ap}(s) + k z_{\rm ap}(s)}, \qquad (4)$$

from which we can determine:

$$G_{\rm ap}(s) = \frac{z_{\rm ap}(s)}{n_{\rm ap}(s)} = \frac{z_{\rm rd}(s)}{k(n_{\rm rd}(s) - z_{\rm rd}(s))}. \qquad (5)$$

$G_{\rm ap}(s)$ is the proposed approximation of $G(s)$. We remark that the order of $G_{\rm ap}(s)$ is $n_{\rm ap} = n_{\rm rd}$, because $n_{\rm rd}(s)$ and $z_{\rm rd}(s)$ are prime. Of course, the approximation of $G(s)$ depends on the approximation-parameter k. It can be seen from the root-locus that the approximation method is continuous in k and that with increasing k the zeros, or equivalently the numerator, are weighted more. Hints for the choice of the approximation-parameter k are given in section 3.

Moreover, for $s = j\omega$ equations (5) and (3) define a bilinear transformation between $H_{\rm rd}(j\omega)$ and $G_{\rm ap}(j\omega)$. Using equation (5), the accuracy of $G_{\rm ap}(j\omega)$ can be computed from the accuracy of $H_{\rm rd}(j\omega)$. Given a bound on the reduction-error $|H(j\omega) - H_{\rm rd}(j\omega)|$ a frequency-dependent bound on the approximation-error $|G(j\omega) - G_{\rm ap}(j\omega)|$ can be derived. In [4] it is shown that the approximation is accurate in the range of the crossover frequency. Thus, the approximation method provides a tool for frequency-weighted reduction [5].

3. The Design Procedure

As mentioned in section 1., the loop-gain $G_0(s)$ should exhibit a certain robustness for stability of the simplified control system. Because of its robustness property (i.e. $\|1 + G_0(j\omega)\| \geq 1 \ \ \forall\omega$) the LQ-method is often used to design a robust loop-gain. Starting from an LQ-optimal loop-gain $G_0(s)$, $G(s)$ is iteratively adjusted in the procedure until the approximated loop-gain $G_{\rm ap}(s)$ meets the desired robustness requirement.

Step 1: LQ-design based on the full-order process model $P(s)$.
To obtain an LQ-optimal loop-gain for the process model $P(s) = \underline{c}'(sI - A)^{-1}\underline{b}$, determine an optimal state feedback $\underline{f_0}$. Thus, we have $G_0(s) = \underline{f_0}'(sI - A)^{-1}\underline{b}$.

Step 2: Approximate the loop-gain $G(s) = G_i(s)$ using the closed-loop reduction method.
Define the closed-loop SIMO system $H(s)$

$$\underline{H}(s) = \begin{bmatrix} \underline{c}' \\ \underline{f}' \end{bmatrix} (sI - A + \underline{b}\underline{f}')^{-1}\underline{b} \qquad (6)$$

Reduce $H(s)$ by the order-reduction method of your choice, e.g. by Moore's method, and let $H_{\rm rd}(s)$ be the reduced system of order $n_{\rm rd}$:

$$\underline{H}_{\rm rd}(s) = \begin{bmatrix} \underline{c}'_{\rm rd} \\ \underline{f}'_{\rm rd} \end{bmatrix} (sI - A_{\rm rd})^{-1}\underline{b}_{\rm rd} \qquad (7)$$

From $H_{\rm rd}(s)$ determine the approximated loop-gain $G_{\rm ap}(s)$ and the approximated process model $P_{\rm ap}(s)$; i.e. according to (5) compute:

$$G_{\rm ap}(s) = \underline{f}'_{\rm ap}(sI - A_{\rm ap})^{-1}\underline{b}_{\rm ap} \qquad (8)$$

$$P_{\rm ap}(s) = \underline{c}'_{\rm ap}(sI - A_{\rm ap})^{-1}\underline{b}_{\rm ap} \qquad (9)$$

where $\underline{b}_{\rm ap} = \underline{b}_{\rm rd}$, $\underline{f}_{\rm ap} = \underline{f}_{\rm rd}$, $\underline{c}_{\rm ap} = \underline{c}_{\rm rd}$, and $A_{\rm ap} = A_{\rm rd} + \underline{b}_{\rm ap}\underline{f}'_{\rm ap}$

Step 3: Iteration: $\underline{f}_{i+1} = \mu\underline{f}_i$, with $\mu > 1$.

3.1 Check the robustness of the approximated loop-gain $G_{\rm ap}(s)$; especially check its phase margin.

3.2 If the phase margin is large enough (e.g. $\geq 60^o$) then goto Step 4, else improve the approximation of the phase by tightening the feedback, i.e. set $f_{i+1} = \mu f_i$ with $\mu > 1$, and go back to Step 2.
Note that by setting $f_{i+1} = \mu f_i$, the corresponding loop-gain $G_{i+1}(s) = \mu G_i(s)$ remains optimal, since for $\mu \geq 1$ the following implication holds:
$|1 + G_i(j\omega)| \geq 1 \ \ \forall\omega \quad \Rightarrow \quad |1 + \mu G_i(j\omega)| \geq 1 \ \ \forall\omega$

3.3 If the loop-gain $G_i(s)$ must not be increased further (due to restriction on the control signal or the presence of measurement noise) an improved approximation of the loop-gain can be achieved by increasing the order of $G_{\rm ap}(s)$; i.e. increase $n_{\rm rd}$ in Step 2.

Step 4: Loop-gain recovery.
Determine the controller $C(s)$ such that the loop-gain is preserved:

$$G_{\rm ap}(s) = P_{\rm ap}(s)C(s) \qquad (10)$$

With $G_{\mathrm{ap}}(s) = z_{\mathrm{ap}}(s)/n_{\mathrm{ap}}(s)$ and $P_{\mathrm{ap}}(s) = y_{\mathrm{ap}}(s)/n_{\mathrm{ap}}(s)$

$$C(s) = P_{\mathrm{ap}}^{-1}(s)G_{\mathrm{ap}}(s) = z_{\mathrm{ap}}(s)/y_{\mathrm{ap}}(s) \qquad (11)$$

Note that almost all order-reduction methods, in particular Moore's method, yield a generic reduced system, i.e. we have degree$[z_{\mathrm{ap}}(s)]$ = degree$[y_{\mathrm{ap}}(s)] = n_{\mathrm{rd}} - 1$. Thus, the controller $C(s)$ is realizable.

Step 5: Simulation
Simulation of the simplified control system completes the procedure.

Due to space constraints we omit examples and we refer to [4] for a more detailed discussion of the method.

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