Partial-order reduction of observers for linear systems

Silvano Balemi [*]

[*] *University of Applied Sciences of Southern Switzerland, Lugano-Manno, Switzerland, (e-mail: silvano.balemi@supsi.ch).*

Abstract: Full-state observers for linear systems use available measurements for the estimation of the entire state of a system. Reduced-order observers instead deliver an estimate only in the unmeasured state subspace while the state values in the measured subspace are taken directly from the measurements. This paper presents a combination of both types of observers which directly uses only part of the measured subspace and estimates/filters the rest of the state space. The order of this new observer is freely selectable between that of the full-state observer and of the reduced-order observer. A practical example demonstrates the validity of the new observer form. The results show that the performance of a state-feedback controller with the new observer is similar to that with the full-state observer.

1. INTRODUCTION

Estimators use a model to reconstruct the entire state of a system using measurements of its inputs and outputs.

Full-state observers estimate the whole state: they filter the available measurements and remove noise present in them (see Kalman and Bucy (1961) and Kailath (1980)). If the measurements contain no practically relevant noise, such an observer implies redundancy because it estimates states (or projections on state subspaces) which are already available from the measurements. Reduced-order observers introduced by Luenberger (1964) allow estimating only the state subspace which is not known from measurements.

In practical applications one would have to choose between the better filtering of measurements thanks to a full-state observer and the lower complexity and higher robustness of controllers using a reduced-order observer. In some cases, neither solution is really satisfactory, because the presence of noise may require filtering while the decreased robustness and added complexity needed to filter all measurements may be unacceptable.

The idea proposed in this paper is to filter only those measurements which are really too noisy, while exploiting directly for control the other measurements which are sufficiently good. Then, the necessary filtering is done with a limited complexity of the estimator.

The result is a reduced-order observer with order between that of the full-state observer (i.e. of the linear system whose states have to be estimated) and that of the reduced-order observer.

2. PARTIAL-ORDER REDUCTION OF OBSERVERS

Given is a linear time-invariant system

$$\begin{cases} \dot{x} = A\cdot x + B\cdot u \\ y = C\cdot x + D\cdot u \end{cases} \tag{1}$$

of order n with the pair (C, A) being observable. The output matrix C is assumed to be full rank. This corresponds to the condition that all measurements are linearly independent.

While restricting at first the analysis to the case $D = 0$, suppose that the outputs y can be partitioned into some "clean" outputs y_c followed by other noisy outputs y_n. Then, the system representation with this partition is given by

$$\begin{cases} \dot{x} = A\cdot x + B\cdot u \\ \begin{bmatrix} y_c \\ y_n \end{bmatrix} = \begin{bmatrix} C_c \\ C_n \end{bmatrix}\cdot x \end{cases} \tag{2}$$

If the outputs do not correspond to measured states, a state transformation

$$P = \begin{bmatrix} C_c \\ C_n \\ T \end{bmatrix} \tag{3}$$

can be applied (the matrix T is chosen in order to make P square and full rank). Then, with the transformation matrix P a new state-space representation with the state vector

$$P\cdot x = \begin{bmatrix} C_c \\ C_n \\ T \end{bmatrix}\cdot x = \begin{bmatrix} C_c\cdot x \\ C_n\cdot x \\ T\cdot x \end{bmatrix} = \begin{bmatrix} y_c \\ y_n \\ w \end{bmatrix} \tag{4}$$

is obtained, in which the measurements are states of the system. The variable w in the state vector indicates the unmeasured state subspace. With the state transformation above, the LTI system (2) can be rewritten as

$$\begin{cases} \begin{bmatrix} \dot{y_c} \\ \dot{y_n} \\ \dot{w} \end{bmatrix} = P\cdot A\cdot P^{-1}\cdot \begin{bmatrix} y_c \\ y_n \\ w \end{bmatrix} + P\cdot B\cdot u \\ y = C\cdot P^{-1}\cdot \begin{bmatrix} y_c \\ y_n \\ w \end{bmatrix} \end{cases}$$

and the resulting state-space matrices partitioned as follows

$$\begin{cases} \begin{bmatrix} \dot{y_c} \\ \dot{y_n} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A_{c,c} & A_{c,n} & A_{c,w} \\ A_{n,c} & A_{n,n} & A_{n,w} \\ A_{w,c} & A_{w,n} & A_{w,w} \end{bmatrix} \cdot \begin{bmatrix} y_c \\ y_n \\ w \end{bmatrix} \\ \qquad + \begin{bmatrix} B_c \\ B_n \\ B_w \end{bmatrix} \cdot u \\ y = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \cdot \begin{bmatrix} y_c \\ y_n \\ w \end{bmatrix} \end{cases} \tag{5}$$

The sizes of the submatrices can be inferred from the sizes of y_c, y_n and w. With the definition of the matrices

$$A_{cn,nw} = \begin{bmatrix} A_{c,n} & A_{c,w} \\ A_{n,n} & A_{n,w} \end{bmatrix}$$
$$A_{cn,c} = \begin{bmatrix} A_{c,c} \\ A_{n,c} \end{bmatrix}$$
$$A_{nw,c} = \begin{bmatrix} A_{n,c} \\ A_{w,c} \end{bmatrix}$$
$$A_{nw,n} = \begin{bmatrix} A_{n,n} \\ A_{w,n} \end{bmatrix}$$
$$A_{nw,w} = \begin{bmatrix} A_{n,w} \\ A_{w,w} \end{bmatrix}$$
$$B_{cn} = \begin{bmatrix} B_c \\ B_n \end{bmatrix}$$
$$B_{nw} = \begin{bmatrix} B_n \\ B_w \end{bmatrix}$$

and with the interpretation of y_c and y_n as known quantities (i.e. as inputs), the system equations can be rearranged as

$$\begin{cases} \begin{bmatrix} \dot{y_n} \\ \dot{w} \end{bmatrix} = [A_{nw,n} - M, A_{nw,w}] \cdot \begin{bmatrix} y_n \\ w \end{bmatrix} \\ \qquad + [\, B_{nw} \;\; A_{nw,c} \;\; M \,] \cdot \begin{bmatrix} u \\ y_c \\ y_n \end{bmatrix} \\ y_r = \begin{bmatrix} A_{cn,nw} \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} y_n \\ w \end{bmatrix} \\ \qquad + \begin{bmatrix} B_{cn} & A_{cn,c} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \cdot \begin{bmatrix} u \\ y_c \\ y_n \end{bmatrix} \end{cases} \tag{6}$$

where

$$y_r = \begin{bmatrix} \dot{y_c} \\ \dot{y_n} \\ y_c \\ y_n \end{bmatrix} \tag{7}$$

The matrix M is a degree of freedom in the representation, because the contribution of y_n to $[\dot{y}_n;\ \dot{w}]$ can be arbitrarily interpreted as coming from a state or from an input. In fact, the first equation of the rearranged form is

$$\begin{bmatrix} \dot{y_n} \\ \dot{w} \end{bmatrix} = (A_{nw,n} - M) \cdot y_n + A_{nw,w} \cdot w + B_{nw} \cdot u$$
$$+ A_{nw,c} \cdot y_c + M \cdot y_n$$
$$= A_{nw,n} \cdot y_n + A_{nw,w} \cdot w + B_{nw} \cdot u + A_{nw,c} \cdot y_c.$$

The observer for this system is given by the equations:

$$\begin{cases} \begin{bmatrix} \dot{\hat{y}}_n \\ \dot{\hat{w}} \end{bmatrix} = [A_{nw,n} - M, A_{nw,w}] \cdot \begin{bmatrix} \hat{y}_n \\ \hat{w} \end{bmatrix} \\ \qquad + [\, B_{nw} \;\; A_{nw,c} \;\; M \,] \cdot \begin{bmatrix} u \\ y_c \\ y_n \end{bmatrix} \\ \qquad + L \cdot (y_r - \hat{y}_r) \\ \hat{y}_r = \begin{bmatrix} A_{cn,nw} \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{y}_n \\ \hat{w} \end{bmatrix} \\ \qquad + \begin{bmatrix} B_{cn} & A_{cn,c} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \cdot \begin{bmatrix} u \\ y_c \\ y_n \end{bmatrix} \end{cases} \tag{8}$$

By substracting the first equation of (8) from the first equation of (6) and by using the following simplification

$$L \cdot (y_r - \hat{y}_r) = L \cdot \begin{bmatrix} A_{cn,nw} \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} y_n - \hat{y}_n \\ w - \hat{w} \end{bmatrix}$$
$$= L_{cn} \cdot A_{cn,nw} \cdot \begin{bmatrix} y_n - \hat{y}_n \\ w - \hat{w} \end{bmatrix} \tag{9}$$

where $L = [L_c,\ L_n,\ \tilde{L}_c,\ \tilde{L}_n]$ and $L_{cn} = [L_c,\ L_n]$ according to the partition of y_r and $\hat{y}_r$, the evolution of the error

$$\begin{bmatrix} e_{y_n} \\ e_w \end{bmatrix} = \begin{bmatrix} y_n - \hat{y}_n \\ w - \hat{w} \end{bmatrix}$$

leads to the error system

$$\begin{bmatrix} \dot{e}_{y_n} \\ \dot{e}_w \end{bmatrix} = ([\, A_{nw,n} - M,\ A_{nw,w} \,] - L_{cn} \cdot A_{cn,nw}) \cdot \begin{bmatrix} e_{y_n} \\ e_w \end{bmatrix} \tag{10}$$

The dynamics of the error system can be arbitrarily chosen if the pair

$$(A_{cn,nw}, [A_{nw,n} - M,\ A_{nw,w}]) \tag{11}$$

is observable. This pair is observable if the matrix M partitioned according to the sizes of y_n and w in

$$M = \begin{bmatrix} M_n \\ M_w \end{bmatrix}$$

is such that $M_w = A_{w,n}$, M_n has its spectrum disjoint from the spectrum of $-A_{w,w}$ and the pair $(A_{cn,n},\ M_n)$ is observable (see proof in the appendix).

Then a matrix L_{cn} can be found to arbitrarily place the poles of the above error system using standard methods.

Unfortunately the observer computation (8) and (9) depends on the signal w, which is not known. In order to circumvent this problem the variable transformation

$$v = \begin{bmatrix} \hat{y}_n \\ \hat{w} \end{bmatrix} - L_{cn} \cdot \begin{bmatrix} y_c \\ y_n \end{bmatrix} \tag{12}$$

is introduced. The first equation of the observer (8) then becomes

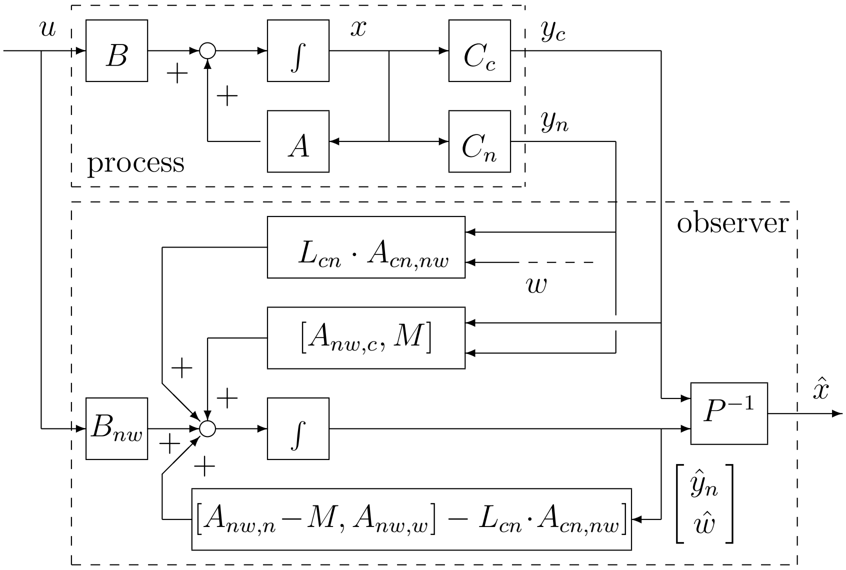

Fig. 1. Block diagram of the reduced-order observer: the state w is not available!

$$
\begin{aligned}
&\dot{v} + L_{cn} \cdot \begin{bmatrix} \dot{y}_c \\ \dot{y}_n \end{bmatrix} \\
&= [\, A_{nw,n} - M,\ A_{nw,w} \,] \cdot \left(v + L_{cn} \cdot \begin{bmatrix} y_c \\ y_n \end{bmatrix} \right) \\
&+ [\, B_{nw}\ \ A_{nw,c}\ \ M \,] \cdot \begin{bmatrix} u \\ y_c \\ y_n \end{bmatrix} \\
&+ L_{cn} \cdot A_{cn,nw} \cdot \left[\begin{bmatrix} y_n \\ w \end{bmatrix} - \left(v + L_{cn} \cdot \begin{bmatrix} y_c \\ y_n \end{bmatrix} \right) \right]
\end{aligned}
$$

With the replacement of

$$
\begin{bmatrix} \dot{y}_c \\ \dot{y}_n \end{bmatrix} = A_{cn,nw} \cdot \begin{bmatrix} y_n \\ w \end{bmatrix} + [B_{cn}\ \ A_{cn,c}\ \ 0] \cdot \begin{bmatrix} u \\ y_c \\ y_n \end{bmatrix}
$$

from (6) into the previous equation and with

$$
y = \begin{bmatrix} y_c \\ y_n \end{bmatrix}
$$

the final expression for the observer becomes

$$
\begin{cases}
\dot{v} = ([\, A_{nw,n} - M,\ A_{nw,w} \,] - L_{cn} \cdot A_{cn,nw}) \cdot v \\
\qquad + (B_{nw} - L_{cn} \cdot B_{cn}) \cdot u \\
\quad + (-L_{cn} \cdot [A_{cn,c},\ 0] - L_{cn} \cdot A_{cn,nw} \cdot L_{cn} \\
\qquad + [A_{nw,c},\ M] \\
\qquad + [A_{nw,n} - M,\ A_{nw,w}] \cdot L_{cn}) \cdot y \\
\hat{x} = P^{-1} \cdot \begin{bmatrix} y_c \\ v + L_{cn} \cdot y \end{bmatrix} = P^{-1} \cdot \begin{bmatrix} \hat{y}_c \\ \hat{y}_n \\ \hat{w} \end{bmatrix}
\end{cases}
\tag{13}
$$

3. OBSERVER IN STANDARD STATE-SPACE FORM

When the matrix D in the state-space representation (1) is non-zero, the observer can be found by simply replacing y with $y - D \cdot u$ in equation (13).

Finally, the observer can be rearranged to take the form of the state-space representation

$$
\begin{cases}
\dot{v} = \hat{A} \cdot v + \hat{B} \cdot \begin{bmatrix} u \\ y \end{bmatrix} \\
\hat{x} = \hat{C} \cdot v + \hat{D} \cdot \begin{bmatrix} u \\ y \end{bmatrix}
\end{cases}
$$

where

$$
\begin{aligned}
\hat{A} &= [\, A_{nw,n} - M,\ A_{nw,w} \,] - L_{cn} \cdot A_{cn,nw} \\
\hat{B} &= [\hat{B}_u,\ \hat{B}_{y_c},\ \hat{B}_{y_n}] \cdot \begin{bmatrix} I & 0 \\ -D & I \end{bmatrix} \\
\hat{C} &= P^{-1} \cdot \begin{bmatrix} 0 \\ I \end{bmatrix} \\
\hat{D} &= P^{-1} \cdot \begin{bmatrix} 0 & [I,\ 0] \\ 0 & L_{cn} \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ -D & I \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{B}_u &= B_{nw} - L_{cn} \cdot B_{cn} \\
\hat{B}_{y_c} &= -L_{cn} \cdot A_{cn,c} - L_{cn} \cdot A_{cn,nw} \cdot L_c + A_{nw,c} \\
&\quad + (A_{nw,n} - M,\ A_{nw,w}) \cdot L_c \\
\hat{B}_{y_n} &= -L_{cn} \cdot A_{cn,nw} \cdot L_n + M \\
&\quad + (A_{nw,n} - M,\ A_{nw,w}) \cdot L_n
\end{aligned}
$$

The discrete-time form of the result is similar to the continuous-time one with the only difference of the substitutions $u(t) \to u_k$, $y(t) \to y_k$, $v(t) \to v_k$ and $\dot{v}(t) \to v_{k+1}$ and the appropriate choice of the observer poles.

4. EXAMPLES

4.1 A synthetic example

Given is the system

$$
\begin{cases}
\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot u \\
y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot x
\end{cases}
\tag{14}
$$

Fig. 2. Inverted pendulum

The poles of this system are at -1 and at $e^{\pm 2\cdot j\cdot \pi/3}$. Suppose the first output is clean and the second output is noisy. Then, according to the conditions on $M = [M_n;\ M_w]$, we choose $M_w = A_{w,n} = 0$, and $M_n = -1$ (no common eigenvalue between M_n and $-A_{w,w} = 1$ and the pair $(A_{cn,n},\ M_n) = ([1; -1], -1))$ is observable).

The resulting error system (10) is given by

$$\begin{bmatrix} \dot{e}_{y_n} \\ \dot{e}_w \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - L_{cn} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} e_{y_n} \\ e_w \end{bmatrix}$$

Then, for a choice of the observer poles at -2, a corresponding matrix L_{cn} can be found with the help of standard methods. One such choice is

$$L_{cn} = \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}$$

which gives the observer

$$\begin{cases} \dot{v} = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} \cdot v + \begin{bmatrix} 0 & -9 & -1 \\ 1 & -4 & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ y \end{bmatrix} \\ \hat{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot v + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ y \end{bmatrix} \end{cases}$$

As expected, the poles of the observer are both at -2. Note that the observer is a second order linear system, while a full-state observer is a third order one and a standard reduced-order observer a first order one.

4.2 The inverted pendulum

The real-life example presented here is the classical inverted pendulum used for teaching in the SUPSI mechatronics laboratory (see Figure 2). A commercial DC motor from Maxon (RE40 model with 1048 pulse encoder) actuates the cart. A 500 pulse quadratic encoder measures the value of the pole angle. The current of the motor is controlled by a driver from Maxon (EPOS P 24/5).

The quantization noise of the input current (1mA) and of the cart position measurement (1.5mm) is irrelevant from a practical point of view. However, the coarse pole angle quantization (3.1mrad) strongly limits the performance of the plant.

In the following, the behavior of the system controlled with the same state-feedback controller in combination with different observer designs is analyzed.

The first design is for a reduced-state observer based on one single measurement: the cart position x. All other observer implementations use both measurements of the pole angle α and of the cart position x.

The second design is for a reduced-state observer, the third design for the partial-order observer proposed in this paper and the fourth and last design is for a full-state observer. The choices for the designs are presented in Table 1: the observer poles are placed at a distance of 5 times the spectral radius of the plant, the argument difference to the closest pole is always $\frac{1}{8}\pi$.

#	observer type	measur.	order	observer poles
1	reduced-order	x	3	$-r,\ r \cdot e^{\pm i \frac{7}{8} \cdot \pi}$
2	reduced-order	$x,\ \alpha$	2	$r \cdot e^{\pm i \frac{15}{16} \cdot \pi}$
3	partial-order	$x,\ \alpha$	3	$-r,\ r \cdot e^{\pm i \frac{7}{8} \cdot \pi}$
4	full-state	$x,\ \alpha$	4	$r \cdot e^{\pm i \frac{13}{16} \cdot \pi},\ r \cdot e^{\pm i \frac{15}{16} \cdot \pi}$

Table 1. Various observer designs. The parameter r is 5 times the spectral radius of the plant.

The simulated trajectories of the controlled cart for different implementations of the observer are shown in Figure 3. The RMS of the angle, position and actuation current for the different implementations are shown in Table 2.

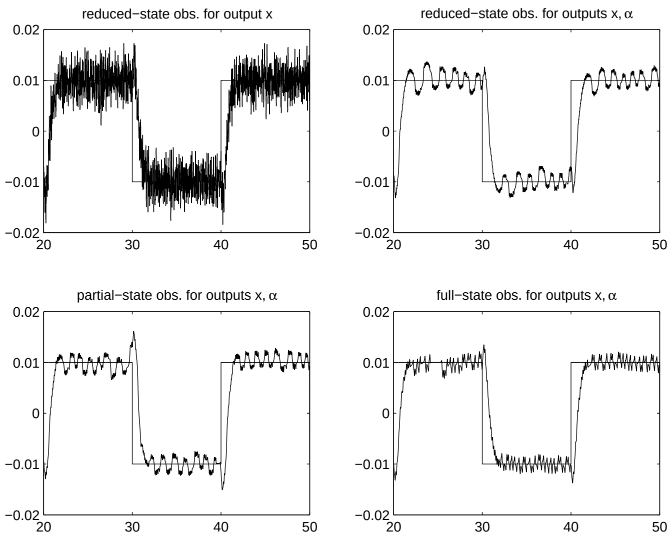

Fig. 3. Trajectory of the controlled cart for various implementations of the observer (implementations #1 to#4 from left to right and then from top to bottom)

Implementation #1 relying only on the cart position measurement is the most critical one with strong high frequency movements. The implementation is not working in practice, the reason being the huge variations in the actuation current which exceeds the 2A current limits of the motor driver.

#	observer type	x_{RMS} [m]	α_{RMS} [mrad]	I_{RMS} [mA]
1	reduced-order	0.0104	3.99	8790.9
2	reduced-order	0.0103	1.35	479.1
3	partial-order	0.0098	1.35	285.5
4	full-state	0.0098	1.20	225.8

Table 2. Comparison of the RMS variations at steady-state of pole angle and cart position for different observers

Implementation #2 using two measurements is well stable but a limit cycle appears because of the angle quantization. In real measurements, the controller with the reduced-order observer is problematic because the bending mode of the pole is excited and the whole system start vibrating.

The full-state implementation #4 presents the best behavior. With the same structure, a Kalman filter minimizing the steady-state error covariance shows a smaller RMS value of the cart movement (0.0088m) at the expense of a slightly larger current (319mA). For the purpose of the comparisons in this papers however, the Kalman filter is not very useful.

The partial-order implementation #3 of the observer takes a position between that of implementations#2 and #4, when considering both the order and the RMS of the position and of the actuation current in simulation.

Measurements on the real pendulum show in all cases a strong limit cycle due to the angle quantization measurement. With the design above, the partial-order observer directly exploiting the cart position measurement seems to work even better than the controller with full-state observer (see Figure 4).

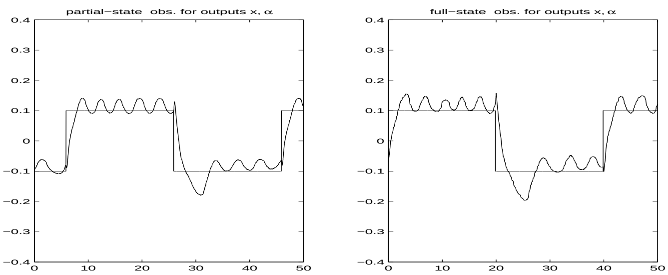

Fig. 4. Measurement of the trajectory of the controlled cart the partial-order observer (left) and for the full-state observer (right).

5. CONCLUSIONS

The paper presented a new version of observers for linear systems with an arbitrarily selectable order between that of the full-state and of the reduced-order observers introduced by Luenberger. The result is particularly useful in practical applications when some measurements must be filtered because they are noisy while the complexity of a full-state observer is undesirable.

Futher work consists in relaxing the constraints on the matrix M and in finding criteria for its choice.

A Matlab script for the computation of the reduced-order observer presented in this article can be downloaded from the author's web page.

APPENDIX

Proposition 1. If the pair (C, A) from (1) is observable, then the pair

$$(A_{cn,nw}, [A_{nw,n} - M,\ A_{nw,w}])$$

from (11) is also observable with

$$M = \begin{bmatrix} M_n \\ M_w \end{bmatrix}$$

such that

- $M_w = A_{w,n}$,
- M_n has its spectrum disjoint from the spectrum of $-A_{w,w}$
- the pair $(A_{cn,n},\ M_n)$ is observable

Proof: First remind that a pair $(C,\ A)$ is observable if and only if

$$\begin{bmatrix} s - A \\ C \end{bmatrix}$$

is full rank for all s. Thus the pair from (11) is observable if and only if

$$\begin{bmatrix} s - A_{n,n} + M_n & -A_{n,w} \\ -A_{w,n} + M_w & s - A_{w,w} \\ A_{c,n} & A_{c,w} \\ A_{n,n} & A_{n,w} \end{bmatrix}$$

is full rank for all s, or also (with the last row added to the first row and by using the choice $M_w = A_{w,n}$) the pair from (11) is observable if and only if

$$\begin{bmatrix} s + M_n & 0 \\ 0 & s - A_{w,w} \\ A_{c,n} & A_{c,w} \\ A_{n,n} & A_{n,w} \end{bmatrix} \tag{15}$$

is full rank for all s,

Note that if the original system (1) is observable, then also the system in the representation (5) is observable and thus

$$\begin{bmatrix} s - A_{c,c} & -A_{c,n} & -A_{c,w} \\ -A_{n,c} & s - A_{n,n} & -A_{n,w} \\ -A_{w,c} & -A_{w,n} & s - A_{w,w} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

is full rank for all s and

$$\begin{bmatrix} -A_{c,w} \\ -A_{n,w} \\ s - A_{w,w} \end{bmatrix}$$

is also full rank for all s. Therefore the second block column of the matrix (15) is full rank for all s. Since by choice of M_n the pair $(A_{cn,n},\ M_n)$ is observable, also the first column of the matrix (15) is full rank for all s.

The only way for the matrix matrix (15) to loose rank is by loosing rank in the first block column for a value of s corresponding to an eigenvalue of M_n but this is impossible because the spectra of M_n and $-A_{w,w}$ are disjoint by choice.

Therefore the matrix (15) is full rank for all s and the pair (11) is observable. ∎

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